Lesson 23 Geometric Series

I. Sigma (Σ)/Summation Notation
II. Geometric Series
   A. Form
   B. Sum
   C. Examples

Reminder - Quiz Fri (10/27) - 1st-order linear differential equations

I. Sigma/Summation Notation

$\boxed{\text{Ex 1}}$ Evaluate

add results together → $\sum_{n=1}^{4} n^2 = 1+4+9+16 = 30$ formula

Plug in integers from 1 to 4

$$\underset{n=1}{1^2} + \underset{n=2}{2^2} + \underset{n=3}{3^2} + \underset{n=4}{4^2} = 30$$

$\boxed{\text{Ex}}$ Use summation notation to write the sum in compact form

$$2+4+6+8+ \cdots +50$$

n	1	2	3	4	.....	n	$\boxed{25}$
nth term	2	4	6	8		2n	50

Ans: $\sum_{n=1}^{25} 2n$

Today, we start to explore infinite sums (Series)

$\boxed{Ex}$ Write the sum in compact form using summation notation.

a) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ ← sign of infinite sum

n	1	2	3	4	...	n
n^{th} term	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$		$\frac{1}{n}$

Ans: $\sum_{n=1}^{\infty} \frac{1}{n}$

b) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

alternating series

n	1	2	3	4		n
n^{th} term	1	$-\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{4}$		$(-1)^{n+1}\frac{1}{n}$

Alternating: $(-1)^n$ or $(-1)^{n+1}$ / $(-1)^{n-1}$

Key: $(-1)^{odd} = -1$

$(-1)^{even} = 1$

Ans: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

c) $\underset{n=0}{\frac{1}{3}} + \underset{n=1}{\frac{1}{6}} + \underset{n=2}{\frac{1}{12}} + \underset{n=3}{\frac{1}{24}} + \underset{n=4}{\frac{1}{48}} + \cdots$ n

$*\frac{1}{2}$ $*\frac{1}{2}$ $*\frac{1}{2}$ $*\frac{1}{2}$

geometric series
You multiply by same ratio every time to get to next term

first term $= \frac{1}{3}$
ratio $= \frac{1}{2}$

n	0	1	2	3	$\cdots$	n
nth term	$\frac{1}{3}$	$\frac{1}{6}=\frac{1}{3}\left(\frac{1}{2}\right)^1$	$\frac{1}{12}=\frac{1}{3}\left(\frac{1}{2}\right)^2$	$\frac{1}{24}=\frac{1}{3}\left(\frac{1}{2}\right)^3$		$\frac{1}{3}\left(\frac{1}{2}\right)^n$

$=\frac{1}{3}\left(\frac{1}{2}\right)^0$

Ex Write the first 5 terms of the series

$$\sum_{n=0}^{\infty}\frac{3^n}{n!} = \frac{3^0}{0!}+\frac{3^1}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots$$

$$= 1+3+\frac{9}{2}+\frac{27}{6}+\frac{81}{24}+\cdots$$

$n!$ only defined for nonnegative integers

$0! = 1$
$1! = 1$
$2! = 1\cdot 2 = 2$
$3! = 1\cdot 2\cdot 3 = 6$
$4! = 1\cdot 2\cdot 3\cdot 4 = 24$

$n! = 1\cdot 2\cdot 3\cdot \cdots \cdot n$

II. Geometric Series

A. Form

(D) A series of the form

$$\underbrace{a+ar}_{\times r}\underbrace{+ar^2}_{\times r}\underbrace{+ar^3}_{\times r}+\cdots = \sum_{n=0}^{\infty} ar^n$$

is called a geometric series

a – 1st term
r – common ratio

$\boxed{Ex}$

a) $2 + \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81} + \cdots$

is geometric $\quad a = 2$

$r = \frac{1}{3}$

b) $1 + \overset{\times \frac{1}{2}}{\frac{1}{2}} + \overset{\times 2/3}{\frac{1}{3}} + \frac{1}{4} + \cdots$

is NOT geometric $\quad \frac{1}{2} \neq \frac{2}{3}$

c) $4 - 1 + \frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \cdots$

is geometric $\quad a = 4$

$r = -\frac{1}{4}$

Big theorem

(T) The geometric series

easier for students than Σ

$$a + ar + ar^2 + ar^3 + \cdots = \sum_{n=0}^{\infty} ar^n$$

- converges to $\frac{a}{1-r}$ → think first / 1-ratio if $|r| < 1$
- diverges if $|r| \geq 1$ ∎

what does $\sum_{n=1}^{\infty} a_n$ mean? $\lim_{k\to\infty} \underbrace{\sum_{n=1}^{k} a_n}_{k\text{th partial sums}}$

Examples

$\boxed{\text{Ex}}$ $\sum_{n=2}^{\infty} \frac{e^{(0.5)n}}{3^{n+2}} = \frac{e^{(0.5)2}}{3^4} + \frac{e^{(0.5)3}}{3^5} + \frac{e^{(0.5)4}}{3^6} + \cdots$

$= \frac{e/81}{1 - \frac{e^{0.5}}{3}}$

geometric $a = \frac{e^1}{3^4} = \frac{e}{81}$

$r = \frac{e^{0.5}}{3}$ $|r| < 1$

$\boxed{\text{Ex}}$ $\sum_{n=1}^{\infty} \left[\left(\frac{2}{3}\right)^n + \left(\frac{1}{4}\right)^{n+1} \right]$

If $\sum \left(\frac{2}{3}\right)^n$ converges and $\sum \left(\frac{1}{4}\right)^{n+1}$ converges, then we can add their answers to get the ans for this problem

$\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots = \frac{2/3}{1 - 2/3} = 2$

$a = 2/3 \quad r = 2/3$

$\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^{n+1} = \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \cdots = \frac{1/16}{1 - 1/4} = \frac{1}{16} \cdot \frac{4}{3}$

$= \frac{1}{12}$

$a = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

$r = \frac{1}{4}$

$\boxed{\text{Ans: } 2 + \frac{1}{12}}$

$\boxed{\text{Ex}}$ Use summation notation to write the number $4.\overline{25}$.

$$4.\overline{25} = 4.25252525 \cdots\cdots$$

$$= 4 + 0.25 + 0.0025 + 0.000025 + \cdots$$

$$= 4 + \frac{25}{100} + \frac{25}{10{,}000} + \frac{25}{1{,}000{,}000} + \cdots$$

$$= 4 + \underbrace{\frac{25}{10^2} + \frac{25}{10^4} + \frac{25}{10^6} + \cdots}_{\text{geometric}}$$

geometric $\quad a = \frac{25}{10^2} \quad r = \frac{1}{10^2}$

At least Two possible answers. Both are correct.

$$4.\overline{25} = 4 + \sum_{n=1}^{\infty} \frac{25}{10^{2n}}$$

or

$$4.\overline{25} = 4 + \sum_{n=0}^{\infty} \frac{25}{10^2}\left(\frac{1}{10^2}\right)^n$$